SOME METHODS OF SOLUTION OF NONLINEAR PROBLEMS OF THE MECHANICS OF DEFORMABLE SOLIDS

(NEKOTORYE METODY RESHENIIA NELINEINYKH ZADACH MEKHANIKI DEFORMIRUEMOGO TELA)

PMM Vol.28, № 3, 1964, pp.418-429

S.V. SIMEONOV
(Sofia, Bulgaria)

(Received January 15, 1964)

The study of various cases of states of stress and strain in a deformable solid often reduces to a search for a minimum of some functional. This leads to the solution of equations with potential operators [1].

Such functional can be the total strain energy, the total supplementary deformation work [2], etc. It is convenient to consider this functional in an appropriately chosen Hilbert space. This paper is confined exclusively to equations with potential operators.

Section 1 deals with the increment of a functional. In this connection, certain concepts have been introduced that will be necessary in the sequel. Sections 2 and 3 present and justify two methods of successive approximations for the solution of the functional equations studied. In Section 4, the application of these methods to certain problems of the mechanics of deformable solids has been considered. In Sections 5 and 6 a proof has been given of the convergence of the Bubnov-Gelerkin method and of the method of "partial approximation" for the solution of a certain class of nonlinear differential equations.

1. Let the functional $f(x)$ be given in the Hilbert space H. We will assume that it admits a Gato differential $Df(x, h)$, and that this differential is a linear functional relative to $h \in H_1 \subset H$.

We will introduce the function $\varphi(t) = f(x + th)$, where t is a numerical parameter. Then

$$\Delta f(x, h) = f(x + h) - f(x) = \varphi(1) - \varphi(0) = \varphi'(0) + {}^1/_2\, W(x, h)$$

Hence, introducing the notation $Ax = \operatorname{grad} f(x)$, we obtain

$$\Delta f(x, h) = f(x + h) - f(x) = (Ax, h) + {}^1/_2\, W(x, h) \qquad (1.1)$$

where

$$W(x, h) = \varphi''(\tau) \quad (0 < \tau < 1), \qquad \varphi''(\tau) = (A'(x + \tau h)\, h, h)$$

Therefore, if the Gato derivative $A'(x)$ of the operator A exists, then Expression (1.1) assumes the form (1.2)

$$\Delta f(x, h) = f(x+h) - f(x) = (Ax, h) + {}^1/_2 (A'(\xi) h, h) \qquad \begin{pmatrix}\xi = x+\tau h \\ 0<\tau<1\end{pmatrix}$$

If the functional $f(x)$ is extremal at the point $x^* \in H$, then as is well known [1 and 3], at this point $(Ax^*, h) = Df(x^*, h) = 0$ for any $h \in H_1$. Consequently, the extremal points of the functionl $f(x)$ need only be sought among the solutions of the functional equation

$$Ax = 0 \tag{1.3}$$

Let us investigate the mechanical significance of the functional $W(x, h)$ (i.e. of $(A'(\xi)h, h)$) in the case when $f(x)$ is the total strain energy of the solid. Then the element $x \in H$ expresses the state of deformation of the solid resulting from an external influence $y \in Y$, where Y will also be a Hilbert space, and $h \in H_1$ can be regarded as a possible displacement of the solid.

Let $x \in H$ be the same state equilibrium of the system, and $h \in H_1$ be a given admissible displacement. Then from (1.1) we obtain

$$\Delta f(x, h) = f(x+h) - f(x) = {}^1/_2 W(x, h) \tag{1.4}$$

Hence it is clear that $W(x, h)$ is twice the energy that must be expended in order to impart the displacement h to the solid.

Now we will assume that the external load y is some function of the parameter λ, i.e. $y = y(\lambda)$. We will introduce the following definitions. The whole of the deformable body and of the external loading y will be called the deformable system. We will also say that we have a deformable system with increasing (decreasing) stiffness, if the functional $W(x(\lambda), h)$ increases (decreases) with increase of the parameter λ for an arbitrary value of the element $h \in H_1$ or remains bounded from below (above) its value at $\lambda = 0$. If $W(x(\lambda), h)$ is independent of λ, and thus also of x, we have a linear deformable system.

If stiffness is understood to mean the capacity of the solid to resist deformation, then the above definitions are justified by the following obvious considerations.

Let us assume that $W(x(\lambda), h)$ increases (decreases) when the value of the parameter λ is increased. This means that for large initial values of the loading it will be necessary to expend more (less) energy in order to achieve the additional displacement h than would be necessary with a smaller initial loading.

If we call linear systems the first class of systems, we will speak of systems with nondecreasing stiffness. Similarly, one can introduce systems with nonincreasing stiffness. It should be noted that one and the same solid can for different types of loading be either of the first or of the second class.

Let it be assumed that for a given state of loading $x_0 \in H$ the solid is in stable equilibrium. Then it follows from (1.2) that in a certain sphere D containing point x_0 the following condition holds

$$W(x, h) = (A'(\xi) h, h) \geqslant \gamma^2 (h, h) > 0 \qquad (\xi \in D,\ h \in H_1) \tag{1.5}$$

where γ = const ; i.e. for any $\xi \in D$ the operator $A'(\xi)$, if it exists, is positive definite and hence self-adjoint. Moreover, it also follows that if the unloaded state is one of stable equilibrium, then, for deformable systems with nondecreasing stiffness, the operator $A'(\xi)$ is positive definite for any $\xi \in H$. This also occurs in systems with decreasing stiffness, when $W(x, h)$, in decreasing, tends to a certain positive limit as e.g. in systems with physical nonlinearities for which the material always has a real strength; i.e. $E_s \geqslant c^2$, where E_s is the shear modulus, and

$$c = \text{const} \neq 0.$$

If, however, in a system with decreasing stiffness for some $\lambda = \lambda_0$ $(x = x_0)$ and $h \neq 0$ we have $W(x_0, h) = 0$, then during subsequent loading the following two cases are usually observed: (a) the solid gradually or suddenly goes over into a new state of equilibrium for which again $W(x, h) > 0$; (b) the equilibrium state becomes indefinite, or does not exist at all. The first case is usually encountered in nonlinear problems, and the second in physically linear problems with a horizontal asymptote in the stress-strain diagram. The first case for plates and shells was studied in detail in [4 and 5]. From what has been said, it follows that the properties of the functional $W(x, h)$, or of the operator $A'(\xi)$, characterize the basic mechanical properties of a deformable system.

The expression $A'(x)h$ approximates the difference $A(x + h) - A(x)$ to an accuracy of terms of order greater than $\|h\|$, therefore, Equation

$$A'(x)\, h = \Delta y \tag{1.6}$$

can be considered as the linear analogue of Equation (1.3) for the determination of the increment h as a result of the additional loading Δy of the system above that loading which corresponds to state x.

2. We will study Equation

$$x = x - \alpha B^{-1} A x \tag{2.1}$$

where $\alpha \neq 0$ is an as yet arbitrary coefficient, and B is some positive definite operator. Since Equation $B^{-1}x = 0$ has the unique solution $x = 0$, it is clear that the solutions of Equation (2.1) will also be the solutions (usually generalized) of Equation (1.3), and vice versa. Let us form the recurrence relation

$$x^{(\nu+1)} = x^{(\nu)} - \alpha B^{-1} A x^{(\nu)} \tag{2.2}$$

If this process converges, then, by virtue of what has been said, its limit point point will be a solution of Equation (1.3). It turns out that in many cases it is possible to guarantee convergence by suitably selecting the coefficient α and the operator B. Moreover, the introduction of operator B^{-1} into (2.1) makes it possible to adapt this process to the approximate solution of different types of differential equations. For this purpose it is necessary to choose the operator B^{-1} in such a way that the element $z = B^{-1}x$ satisfies all boundary conditions of the problem for any $x \in H$. This means also that each subsequent approximation may be determined as the solution of the differential equation

$$Bx = Bx^{(\nu)} - \alpha A x^{(\nu)} \tag{2.3}$$

the right-hand side of which is known. Such solution satisfies the boundary conditions.

Additional requirements, which operator B and coefficient α must satisfy, are given by the following theorem.

Theorem 2.1. If the functional $W(x, h)$, the operator B and coefficient α satisfy conditions

$$W(x, h) \leqslant K(Bh, h) \qquad 0 < \alpha < 2/K \qquad (x \in H, h \in H_1) \tag{2.4}$$

where K is positive constant, the process (2.2) always converges to some solution x^* of Equation (1.3) independent of the choice of the initial approximation, if the functional $f(x)$ is bounded from below and increases outside a certain sphere D.

For the case when $W(x, h) \geqslant \gamma^2 \| h \|^2$ for arbitrary $x \in H$ and $h \in H_1$ and $\gamma = \text{const} \neq 0$, the solution of Equation (1.3) is unique.

Note. If the operator A admits a Gato derivative $A'(x)$, then $W(x, h)$ in (2.4) can be replaced by $(A'(x)h, h)$.

Proof. We will say that there is a descent on $f(x)$ if on changing from $x^{(\nu)}$ to $x^{(\nu+1)}$ we have

$$f(x^{(\nu+1)}) - f(x^{(\nu)}) < 0$$

First we will prove that under conditions (2.4) the process (2.2) leads to a descent on $f(x)$. For this we note that $x^{(\nu+1)} = x^{(\nu)} + h^{(\nu)}$ and substitute this expression into (1.1)

$$\Delta f(x^{(\nu)}, h^{(\nu)}) = f(x^{(\nu)} + h^{(\nu)}) - f(x^{(\nu)}) = (Ax^{(\nu)}, h^{(\nu)}) + {}^1/_2 W(x^{(\nu)}, h^{(\nu)}) \tag{2.5}$$

However, from (2.2) it follows that $Ax^{(\nu)} = -\alpha^{-1}Bh^{(\nu)}$. Hence

$$\Delta f(x^{(\nu)}, h^{(\nu)}) = -\alpha^{-1}(Bh^{(\nu)}, h^{(\nu)}) + {}^1/_2 (W(x^{(\nu)}, h^{(\nu)})$$

On the basis of (2.4) there exists a number $\theta_\nu \leqslant 1$ such that

$$W(x^{(\nu)}, h^{(\nu)}) = \theta_\nu K(Bh^{(\nu)}, h^{(\nu)})$$

Consequently

$$\Delta f(x^{(\nu)}, h^{(\nu)}) = (-\alpha^{-1} + {}^1/_2\theta_\nu K)(Bh^{(\nu)}, h^{(\nu)}) \tag{2.6}$$

Then, having in mind the condition $(Bh^{(\nu)}, h^{(\nu)}) > 0$, it is immediately obvious that, on choosing α in the interval (2.4), we will have $\Delta f < 0$, i.e. the process (2.2) leads to a descent on $f(x)$. Hence the sequence $\{f(x^{(\nu)})\}$ is decreasing. Since, according to the condition of the theorem, the sequence is bounded from below, and $f(x)$ increases outside some sphere D, this sequence has a limit point $f(x')$ $(x' \in D)$. We will show that this is the only point $f(x^*)$.

In fact, from the convergence of the sequence $\{f(x^{(\nu)})\}$ it follows that $\Delta f(x^{(\nu)}, h^{(\nu)}) \to 0$, starting with some ν. Thus, the right-hand side of (2.6) converges to zero. However, in so far as $-\alpha^{-1} + {}^1/_2\theta_\nu K \neq 0$, this is possible only provided $(B(x^{(\nu+1)} - x^{(\nu)}), x^{(\nu+1)} - x^{(\nu)}) \to 0$. Since B is a positive definite operator, this means that $x^{(\nu+1)} \to x^{(\nu)}$. Then from (2.2) it follows that $x^{(\nu)} \to x^*$.

Finally, let us assume that $W(x, h) \geqslant \gamma^2 \|h\|^2$ when $x \in H$ and $h \in H_1$. Then $f(x)$ satisfies the conditions of the theorem. We will assume in this case that there are two solutions x_1^* and x_2^* of Equation (1.3), where

$$f(x_2^*) \leqslant f(x_1^*).$$

Writing $x_2^* - x_1^* = h^*$, it then follows from (1.1) that

$$f(x_2^*) - f(x_1^*) = {}^1/_2 W(x_1^*, h^*) > 0$$

which contradicts the above assumption. Consequently $x_1^* = x_2^*$. The theorem has been proved.

N o t e . The recurrence relation (2.2) can be generalized by making α and B depend on ν. However, this improved convergence complicates the computational program. The convergence can also be improved by making use of the results in Section 4.

If the operator $A'(x^{(0)})$, is used as operator B, then the recurrence relation (2.2) with $\alpha = 1$ assumes the form of the recurrence relation of the modified Newton method [6].

In [7] for the solution of equations of type (1.3) use was also made of a recurrence relation of the type (2.2) with α = const and $\alpha = \alpha(\nu)$, but with different assumptions concerning the operators A and B. In particular, it was assumed that $f(x)$ is given in some Banach space and that the solution of Equation (1.3) is unique. The restrictions made here for the initial functional $f(x)$ and the operator A are more general, and the convergence conditions found are more convenient for direct application and study. In this sense, the results obtained can be regarded as further developments of some ideas in the above-mentioned paper [7]. The process of steepest descent (without the use of operator B) for the determination of the extremal point of the functional has been investigated in [8].

3. In place of the functional equation (1.3), let us consider the following system of nonlinear functional equations

$$A_k x = 0 \qquad (k = 1, \ldots, n) \tag{3.1}$$

where A_k is a given operator. Such systems describe many mechanics problems including the basic problem of the equilibrium of an element of a deformable solid, where x is the displacement vector.

We will assume that in the process of successive approximations the ν-th approximation $x^{(\nu)} = (x_1^{(\nu)}, \ldots, x_n^{(\nu)})$ has been reached. As the next approximation we will take the element $x^{(\nu+1)}$ with coordinates

$$x_i^{(\nu+1)} = x_i^{(\nu)}, \qquad x_k^{(\nu+1)} = x_k^{(\nu)} - \alpha_k B_k^{-1} A_k x^{(\nu)}$$
$$(i = 1, \ldots, k-1,\ k+1, \ldots, n) \tag{3.2}$$

where B_k are certain positive definite operators.

This process means that at each step only one coordinate is altered. For the following step either the same operation can be repeated or we can proceed to a new coordinate. By a cycle we will mean a set of a definite number of such operations applied to all coordinates x_k independent of the adopted order concerning index k. By cycling we will mean a sequence of such cycles, the structures of which can be different.

If an increase h_k is imparted to the k-th coordinate alone, then in this case

$$f(x + h_k) - f(x) = (A_k x, h_k) + {}^1/_2\, W_k(x, h_k) \tag{3.3}$$

where $W(x, h_k)$ has the same meaning as in Section 1.

T e o r e m 3.1 . If there exist operators B_k such that

$$W_k(x, h_k) \leqslant K_k(B_k h_k, h_k), \qquad 0 < \alpha_k < 2/K_k \qquad (x \in H,\ h_k \in H_1) \tag{3.4}$$

where K_k are positive constants, then for functionals satisfying the conditions of Theorem 2.1, the process (3.2) always converges to some solution x^* of the system (3.1), independent of form of the cycling adopted and of the nature of the initial approximation.

P r o o f . As in the preceding case, it is not difficult here to prove that uncer condition (3.4) process (3.2) gives rise to a descent on $f(x)$. In fact, since now $h^{(\nu)} = (0, \ldots, 0, h_k^{(\nu)}, 0, \ldots, 0)$, we have

$$f(x^{(\nu+1)}) - f(x^{(\nu)}) = (A_k x^{(\nu)}, h_k^{(\nu)}) + {}^1/_2 W_k(x^{(\nu)}, h_k^{(\nu)}) \tag{3.5}$$

On the basis of (3.4) we hence obtain

$$f(x^{(\nu+1)}) - f(x^{(\nu)}) = -(\alpha_k^{-1} - {}^1/_2\, \theta_{k\nu} K_k)(B_k h_k^{(\nu)}, h_k^{(\nu)}) \qquad (\theta_{k\nu} \leqslant 1) \tag{3.6}$$

Since B_k are positive definite operators, the right-hand side of (3.6) is negative for α_k in the interval (3.4), i.e. there is a descent on $f(x)$. By considerations similar to those in the proof of the preceding theorem, we will deduce that the sequence $\{f(x^{(\nu)})\}$ has a limit point $f(x')$. It is necessary to prove that the element x' can only be one of the solutions of system (3.1). For this purpose we form the difference

$$f(x^{(N)}) - f(x') = \sum_{\nu=N}^{\infty} [f(x^{(\nu)}) - f(x^{(\nu+1)})] = \sum_{\nu=N}^{\infty} \mu_{k\nu}(B_k h^{(\nu)}, h^{(\nu)})$$

$$(\mu_{k\nu} = \alpha_k^{-1} - {}^1/_2\, \theta_{k\nu} K_k > 0)$$

Here $f(x^{(N)})$ is the value of the functional attained after the N-th cycle. Setting $\mu = \min\{\mu_{k\nu}\} \neq 0$, we obtain

$$\sum_{\nu=N}^{\infty} (B_k h^{(\nu)}, h^{(\nu)}) \leqslant \mu^{-1} [f(x^{(N)}) - f(x')]$$

From the convergence of the sequence $\{f(x^{(\nu)})\}$ it follows that to an arbitrary small constant ε_N there corresponds an index N such that

$$f(x^{(N)}) - f(x') \leqslant \varepsilon_N.$$

Consequently

$$\sum_{\nu=N}^{\infty} (B_k h^{(\nu)}, h^{(\nu)}) \leqslant \mu^{-1}\varepsilon_N$$

Whence follows the convergence of the series

$$\sum_{\nu=1}^{\infty} (B_k h^{(\nu)}, h^{(\nu)})$$

Then, $h^{(\nu)} \to 0$, starting with some $\nu = N$, i.e. $x_k^{(\nu+1)} \to x_k^{(\nu)}$ simultaneously for all $k = 1, \ldots, n$. Having this in view, it follows from (3.2) that $x^{(\nu)} \to x^*$. The theorem has been proved.

Now we will consider this same process of successive approximations, but with a different method of finding $x_k^{(\nu+1)}$, which turns out to be very suitable for the solution of many mechanics problems.

We will assume that the point $x^{(\nu)}$ has been reached, and it is necessary to determine $x_k^{(\nu+1)}$. With this aim, we substitute the found values

$$x_1^{(\nu)}, \ldots, x_{k-1}^{(\nu)}, x_{k+1}^{(\nu)}, \ldots, x_n^{(\nu)}$$

into the k-th equation of the system (3.1) and write it down in the form

$$A_{k\nu} x_k = 0 \tag{3.7}$$

By some method we will find an approximate solution of this equation. We will denote it by $x_k^{(\nu+1)}$. We have the following theorem.

Theorem 3.2. Under the conditions of Theorem 3.1, if the subsequent approximations $x_k^{(\nu+1)}$ are determined as approximate solutions of Equation (3.7), then for an arbitrary cycling this process converges to a certain solution of the system (3.1) independent of the choice of the initial approximation.

The proof of this theorem follows directly from Theorems 2.1 and 3.1. In fact, we assume that for arbitrary k the operation (3.2) is repeated an infinite number of times.

Conditions (3.4) correspond to conditions (2.4) of Theorem 2.1. Therefore this process converges to the exact solution of Equation (3.7). If we limit the number of operations, we obtain an approximate solution with any desired degree of accuracy. From the convergence of this process follows the validity of the theorem.

The above method of descent represents a generalization of the Gauss-Seidel solution of linear algebraic equations.

4. We will now turn to the application of the above method to some problems in continuum mechanics. First we will consider the question of the existence of a unique state of equilibrium of a deformable system. However, we will not speak of uniqueness in the general case, but of uniqueness of a state of equilibrium in a class of possible displacements $h \in H_1$, corresponding to the given deformable system. Then from Theorem 2.1 we obtain the following results.

Result 1. If, in order to impart to a deformable system an arbitrary admissible displacement from some possible equilibrium state, it is necessary to deliver positive work, then the system has a unique state of equilibrium for arbitrary values of the parameter λ. Mathematically, this condition can be expressed by the positive definiteness of the operator $A'(x)\,(x \in H)$ in the space $H_1 \subset H$.

It is clear that in systems with increasing stiffness there exists always a unique state of equilibrium in the above sense. The same applies for systems with decreasing stiffness if the functional $W(x, h)$, or $(A'(x)h, h)$, on decreasing, converges to some positive limit, as occurs in physically nonlinear problems in the case where the material exhibits actual hardening. However, this does not mean the exclusion of slip planes. In many cases it is not difficult, on the basis of the above results, to check the admissibility of such a plane and also its length, which guarantees the existence of a unique solution.

Most nonlinear equations of continuum mechanics can be written down as

$$Ax \equiv Bx + Cx = 0 \tag{4.1}$$

where B is an operator of a corresponding linear problem and C is some nonlinear operator. Since the operator B is usually positive definite, it can be employed in the recurrence relation (2.2). Then the latter assumes the form

$$x^{(\nu+1)} = (1 - \alpha)\, x^{(\nu)} - \alpha B^{-1} C x^{(\nu)} \tag{4.2}$$

In many cases it is convenient to include into operator B (4.1) only some of the terms of the operator for the linear problem and all the remaining terms are combined with the aid of operator C, since this can lead to a greatly simplified computational program. For this purpuse it is also possible to assume a quite similar well-known operator that satisfies the conditions of Theorem 2.1 .

Now we will study some special features in the application of process (2.2), or (4.2), to the search for state of equilibrium in the above two types of deformable systems, i.e. with decreasing and increasing stiffness. We will begin with the first type.

For potentials with gradients of type (4.1) it is easy to see that $W(x, h) = (Bh, h) + W_1(x, h)$. Moreover, since $Ax \to Bx$ as $\|x\| \to 0$, then $W(0,h) = (Bh,h)$. In addition, $W(x, h)$ decreases with increase of loading; therefore, $\max W(x, h) = W(0, h)$, and hence it follows that $W_1(x, h) < 0$. Then condition (2.4), which in the present case assumes the form

$$(Bh, h) + W_1(x, h) \leqslant K\,(Bh, h)$$

will always be satisfied, also with $K = 1$. Hence we obtain:

Result 2. For systems with decreasing stiffness, process (4.2) always converges if we take

$$0 < \alpha < 2 \tag{4.3}$$

However, for values of $W(x, h)$ close to zero, i.e. for values

$$W_1(x, h) \to -(Bh, h),$$

the upper boundary of coefficient α quickly increases and for obtaining good convergence it is necessary to take $\alpha > 2$.

From condition (4.3) it is evident that in the problems considered process (4.2) converges also for $\alpha = 1$. Then the recurrence relation (4.2) assumes the form

$$x^{(\nu+1)} = -B^{-1} C x^{(\nu)} \tag{4.4}$$

In the theory of plasticity with proportional loading, process (4.4) corresponds with the method of "elastic solutions" [9]. Its convergence for the first and second boundary-value problems in the case of actual work-hardening materials without slip planes was proved in [10], where the rate of convergence was also demonstrated. We note, however, that in many studies of elastic-plastic deformation the method of coordinate descent turns out to be more effective. In the case when the problem is solved in terms of stresses, this method can be called the method of successive equilibrium or even the method of successive coupling.

More complicated are those systems with decreasing stiffness, which with increase of the loading parameter λ are capable of becoming unstable or going into a state of neutral equilibrium as a consequence of the physical nonlinearity of the material, when the stress-strain diagram has a horizontal asymptote, or a slip plane. Apparently, it is more convenient to reduce investigations of the behavior of the equilibrium state of such systems to the study of the loading process. For this one must introduce the time as a parameter [11 and 12] or treat the loading as a variable quantity for the solution of the system of nonlinear equations obtained by either Ritz or Galerkin method [5]. In both cases the problem is reduced to the solution of a Cauchy problem. Depending on the form of the systems of nonlinear differential equations obtained, this may lead to the accumulation of appreciable errors in the integration process. A similar possibity of carrying out this idea arises in the application of the above-described processes (2.2), (3.2) and (3.7) to the determination of equilibrium states corresponding to different atages of a successive loading. This makes it possible to carry out in principle the calculations to an arbitrary degree of accuracy. It should be noted, however, that when values of $W(x, h)$ close to zero are reached, it is necessary to increase the value of coefficient α at each stage of the loading in order to obtain an acceptable rate of convergence. The occurrence of slow convergence at a fixed value of coefficient α, or divergence of processes (2.2) and (3.2) for the indicated physically nonlinear problems may be interpreted as exhaustion, in some sense, of the actual ability of the construction. These processes of successive approximation also make possible the determination of: (1) the instant of "snap through" in geometrically nonlinear problems of the type encountered in shell theory, and (2) "snap through" stable state of equilibrium since in this case the total energy of the system always remains bounded from below.

The convergence of the successive approximations in the treated problems can be increased, if at each stage of the loading y a new operator is adopted in the corresponding recurrence formulas; this is reduced to the calculation of successive approximations as solutions of the approximately linearized equations

$$B_{(i)} h^{(\nu+1)} = -\alpha_{(i)} A x^{(\nu)}, \quad h^{(\nu+1)} = x^{(\nu+1)} - x^{(\nu)} \tag{4.5}$$

where $B_{(i)}$ is an operator close to the operator $A'(x)$ for x obtained as the solution in the preceding loading stage y_{i-1}.

The matter is radically simplified for systems with nondecreasing stiffness, possesing unique states of equilibrium as a consequence of which a correctly formulated "mathematical model" allows a unique solution. Elastic systems which in the main undergo dilatation during the process of loading should be first of all regarded as the above mentioned systems. In this case the application of process (3.7) does not encounter any major difficulties, and the peculiarity of applying processes (2.2) and (3.2) consists in the fact that one can not limit in advance the coefficient α from the above as it was possible in (4.3). This is because the value of the coefficient c in the inequality $0 < \alpha \leqslant c < 2$ will be the smaller, the higher the value of $W(x, h)$ reached in the course of deformation, i.e. as the stiffness of the deformable system is higher. In this case the coefficient α can be determined in several ways: for example, one can make use of the solution $x_0{}^*$ of the linear problem; in this case $\| x_0{}^* \| > \| x^* \|$, which leads t the value

$$K \geqslant \frac{(A'(x_0{}^*)\, h, h)}{(Bh, h)} \qquad (h \in H_1)$$

i.e. to the smallest value $c = 2/K$.

5. Let Equation (1.3) be a nonlinear differential equation given in a closed region Ω and with region of definition D. We will assume that the solution x^* of this equation can be represented in the form of the convergent series

$$x^*(P) = a_1\varphi_1(x) + \alpha_2\varphi_2(x) + \ldots \tag{5.1}$$

which is differentiable as many times as needed by the differential equations. Here $\varphi_i(P)$ is a complete system of functions satisfying tne boundary conditions, and P is an arbitrary point in region Ω.

As is well known, it follows from the Bubnov-Galerkin method that function $x(P)$ is approximated by means of n terms of series (5.1)

$$x_n(P) = a_1{}^{(n)}\varphi_1(P) + \ldots + a_n{}^{(n)}\varphi_n(P) \tag{5.2}$$

where the coefficients $a_i{}^{(n)}$ are determined from the nonlinear system of equations

$$(Ax_n, \varphi_k) = 0 \qquad (k = 1, \ldots, n) \tag{5.3}$$

which can also be written down as

$$A_1 a^{(n)} = 0 \qquad (k = 1, \ldots, n) \qquad (a^{(n)} = (a_1{}^{(n)}, \ldots, a_n{}^{(n)})) \tag{5.4}$$

T h e o r e m 5.1 . If solution $x^*(P)$ of equations (1.3) can be expanded in the form (5.1) and

$$W(x, h) \geqslant \gamma^2 \| h \|^2 \qquad (x, h \in D) \tag{5.5}$$

then the Bubnov-Galerkin method converges, i.e. $x_n(P) \to x^*(P)$.

P r o o f .. First we prove that, under the conditions of the theorem, system (5.4) has a solution that is unique. We will set

$$h = c_1\varphi_1(P) + \cdots + c_n\varphi_n(P).$$

Then

$$f(x_n + h) - f(x_n) = (Ax_n, h) + {}^1/_2 W(x_n, h) = (A_1 a^{(n)}, c) + {}^1/_2 W_1(a^{(n)}, c)$$

and since $W(a^{(n)}, c) = W(x_n, h) \geqslant \gamma^2 \|h\|^2$, the existence of a unique solution of system (5.4) follows from Theorem 2.1 .

We will further prove that with the coefficients $a_i^{(n)}$ so determined the transition from x_{n-1} to x_n leads to a descent on $f(x)$. For this purpose we here set

$$x_n = x_{n-1} + h_n = x_{n-1} + c_1\varphi_1(P) + \cdots + c_n\varphi_n(P)$$

Then

$$f(x_{n-1}) - f(x_n) = -(Ax_n, h_n) + {}^1/_2 W(x_{n-1}, h_n)$$

However, by virtue of (5.3)

$$(Ax_n, h_n) = \Big(Ax_n, \sum_{i=1}^{n} c_i\varphi_i(P)\Big) = \sum_{i=1}^{n} c_i (Ax_n, \varphi_i) = 0$$

and consequently

$$f(x_{n-1}) - f(x_n) = {}^1/_2 W(x_{n-1}, h_n) > 0$$

i.e. $f(x_{n-1}) > f(x_n)$. Since under condition (5.5) the functional $f(x)$ is bounded from below, the sequence $\{f(x_n)\}$ has a limit point $f(x')$. We will prove that this point coincides with $f(x^*)$. Let

$$h = x^* - x' = c_1\varphi_1(P) + c_2\varphi_2(P) + \cdots + c_n\varphi_n + \cdots$$

We form the difference $f(x^*) - f(x')$, taking into account (5.3) for $n = \infty$

$$f(x^*) - f(x') = (Ax', h) + {}^1/_2 W(x', h) =$$

$$= \Big(Ax', \sum_{i=1}^{\infty} c_i\varphi_i(P)\Big) + {}^1/_2 W(x', h) = {}^1/_2 W(x', h) > 0$$

However, since $f(x')$ cannot be smaller than the minimum value of $f(x^*)$ of the functional $f(x)$, it follows that $h \equiv 0$, i.e. $x^* = x'$. The theorem has been proved.

N o t e . As a consequence of the above theorem we have the convergence of the generalized method of Bubnov-Galerkin, and also of the Ritz method for the problems considered.

In application to problems in the nonlinear theory of shallow shells, the convergence of Bubnov-Galerkin method has been proved in [13 and 14], and in [15] it has been studied in a more general formulation. The convergence of Ritz method with sufficiently general assumption regarding the functional $f(x)$ has recently been proved in [16].

6. Let the following nonlinear differential equation be prescribed in a closed region Ω

$$Ax \equiv Bx + Cx = 0 \tag{6.1}$$

where B is some linear operator such that for the given boundary conditions of the problem it is possible to find the exact solution of the differential equation

$$Bx = y \tag{6.2}$$

where y is an arbitrary function bounded in Ω . We will assume that the unknown function can, with sufficient accuracy, be represented in the form

$$\chi_n(P) = \sum_{i=1}^{n} a_i^{(n)} \varphi_i(P) \tag{6.3}$$

where $a_i^{(n)}$ are certain coefficients and $\varphi_i(P)$ form a complete system of functions. Substituting (6.3) into the nonlinear operator C of Equation (6.1) we obtain

$$Bx_n = -C\Big(\sum_{i=1}^{n} a_i^{(n)} \varphi_i(P)\Big) \tag{6.4}$$

From what was said above about (6.2), Equation (6.4) can be solved exactly. Its solution will have the form

$$x_n = -B^{-1}C\Big(\sum_{i=1}^{n} a_i^{(n)}\varphi_i(P)\Big) = x_n(P, a_1^{(n)}, \ldots, a_n^{(n)}) \tag{6.5}$$

which satisfies all boundary conditions of the problems for arbitrary values of the coefficients $a_i^{(n)}$. In order that (6.5) should be an exact solution of the problem it is necessary and sufficient that in region Ω we have the identity $\chi_n(P) \equiv x_n(P)$. For satisfying this condition, use can be made of the undetermined coefficients $a_i^{(n)}$. The approximation of the method consists in the fact that it is practically impossible to do this exactly.

In the computational treatment the above method is rather more complicated than the Bubnov-Galerkin method, it has, however, certain important advantages over the latter. For example, it enables the solution of partial differential equations with complex boundary conditions, such as occur, e.g. in the solution of a certain class of "contact problems", etc. It is impossible, or at least very difficult, to satisfy these boundary conditions with the Galerkin method. Moreover, in many cases this method leads to a more accurate solution with the same number of coefficients $a_i^{(n)}$. It should also be pointed out that here the functions $\varphi_i(p)$ in (6.3) must not necessarily satisfy all boundary conditions of the problem, as it is required in the application of the Bubnov-Galerkin method.

In combination with the method of successive approximations, this method was first suggested by Novozhilov [17] for the solution of ordinary differential equations, requiring that the functions $\varphi_i(p)$ satisfy all the boundary conditions. Its subsequent development for the solution of those same differential equations was given in [18]. In both works there is no proof of convergence. In the form presented here this method has been applied to the solution of nonlinear particular differential equations.(*) In [19] it has been applied to the solution of ordinary linear differential equations with variable coefficients. In the same work there is also proof of convergence for one class of differential equations; there is also a comparison of its accuracy with that using the Bubnov-Galerkin method.

The identification of the functions $\chi_n(P)$ and $x_n(P)$ in the given region Ω, as is well known [20], can be achieved in several ways. However, a suitable one in the present case consists in orthogonalizing the differences $\chi_n(P) - x_n(P)$ to all functions $\varphi_i(P)$. Then the following system of equations is obtained for the determination of the coefficients $a_i^{(n)}$:

$$\Big(\sum_{i=1}^{n} a_i^{(n)}\varphi_i(P) + B^{-1}C\Big(\sum_{i=1}^{n} a_i^{(n)}\varphi_i(P)\Big),\ \varphi_k(P)\Big) = 0 \qquad (k = 1, \ldots, n) \tag{6.6}$$

*) S.V. Simeonov. Dissertation. Leningrad Ship-building Institute, 1957.

In this case we have the following theorem.

Theorem 6.1. If as functions $\varphi_i(P)$ we choose the eigenfunctions of the selfadjoint operator B, and the coefficients $a_i^{(n)}$ are determined from the system of equations (6.6), then the above method converges, i.e. $x_n(P) \to x^*(P)$, in all cases in which the Bubnov-Galerkin method converges. However, the approximate solution $x_n(P)$ can be more accurate than $\chi_n(P)$ obtained by the Bubnov-Galerkin method only when the method of successive approximations (2.2) applied to equation (6.1) converges with $\alpha = 1$.

Proof. We will prove that system (6.6) is identical with system (5.3) of the Bubnov-Galerkin method. For this purpose we replace Ax_n in (5.3) by the corresponding expression from (6.1), keeping in mind that operator B is selfadjoint, and we obtain

$$(A\chi_n, \varphi_k) = (B\chi_n + C\chi_n, \varphi_k) = (\chi_n + B^{-1}C\chi_n, B\varphi_k)$$

However, since φ_k is an eigenfunction of operator B then

$$(A\chi_n, \varphi_k) = \lambda_k (\chi_n + B^{-1}C\chi_n, \varphi_k) = 0 \qquad (k = 1, \ldots, n)$$

where λ_k are the corresponding eigenvalues. Comparison of this system with system (6.6) makes it evident that they are identical. Consequently, if the Bubnov-Galerkin method converges, then so also does the present method.

From what has been said above it is clear that $\chi_n(P)$ is none other than the approximation obtained by the Bubnov-Galerkin method, and according to (6.4) $x_n(P) = -B^{-1}C\chi_n(P)$. Comparing this expression with (4.2) shows that $x_n(P)$ can be regarded as the next approximation of the successive approximations (4.2) with $\alpha = 1$, if we take $\chi_n(P)$ as initial approximation. Then the convergence of this process is a sufficient condition in order that $x_n(P)$ be more exact than $\chi_n(P)$.

Note. Usually $x_n(P)$ is obtained more exactly by determining x_n not by (6.5) but by Formula

$$x_n(P) = (1 - \alpha)\chi_n(P) - \alpha B^{-1}C\chi_n(P)$$

Here α can be determined according to Section 4, and $\varphi_i(P)$ satisfy all the boundary conditions of the problem.

BIBLIOGRAPHY

1. Vainberg, M.M., Variatsionnye metody issledovaniia nelineinykh operatorov (Variational Methods of Studying Nonlinear Operators). Gostekhizdat, 1956.
2. Novozhilov, V.V., Teoriia uprugosti (Theory of Elasticity). Sudpromgiz, 1958.
3. Liusternik, L.A. and Sobolev, V.I., Elementy funktsional'nogo analiza (Elements of Functional Analysis). Gostekhizdat, 1951.
4. Vorovich, I.I., Nekotorye voprosy ustoichivosti obolochek v bol'shom (Some questions of stability of shells in the large). Dokl.Akad.Nauk SSSR, Vol.122, № 1, 1958.
5. Kurdiumov, A.A., K teorii fizicheski i geometricheski nelineinykh zadach izgiba i ustoichivosti plastin i obolochek (On the theory of physically and geometrically nonlinear problems of bending and stability of plates and shells). Trud.Leningr.korablestr.Inst., № 34, 1961.

6. Kantorovich, L.V. and Akilov, G.P., Funktsional'nyi analiz v normirovannykh prostranstvakh (Functional Analisis in Standardized Spaces). Fizmatgiz, 1959.

7. Vainberg, M.M., O skhodimosti metoda naiskoreishego spuska dlia nelineinykh uravnenii (On the convergence of the method of steepest descent for nonlinear equations). Dokl.Akad.Nauk SSSR, Vol.130, № 1, 1960; Sib.mat.Zh., Vol.2, № 2, 1961.

8. Poliak, B.T., Gradientnye metody minimizatsii funktsionalov (Gradient methods of minimization of functionals). Zh.vychisl.mat. i mat.Fiz., Vol.3, № 4, 1963.

9. Il'iushin, A.A., Plastichnost' (Plasticity). Gostekhizdat, 1948.

10. Vorovich, I.I. and Krasovskii, Iu.P., O metode uprugikh reshenii (On the method of elastic solutions). Dokl.Akad.Nauk SSSR, Vol.126, № 4, 1959.

11. Feodos'ev, V.I., Ob odnom sposobe resheniia nelineinykh zadach ustoichivosti deformiruemykh sistem (A method of solving nonlinear problems of the stability of deformable systems). *PMM* Vol.27, № 2, 1963.

12. Feodos'ev, V.I., Primenenie shagovogo metoda k analizu ustoichivosti szhatogo sterzhnia (Application of a step-by-step method to the analysis of stability of a compressed bar). *PMM* Vol.27, № 5, 1963.

13. Vorovich, I.I., O sushchestvovanii reshenii v nelineinoi teorii obolochek (On the existence of solutions in the nonlinear theory of shells). Izv.Akad.Nauk SSSR, (Ser.matem.), Vol.19, № 4, 1955.

14. Vorovich, I.I., O metode Bubnova-Galerkina v nelineinoi teorii kolebanii pologikh obolochek (On the method of Bubnov-Galerkin in the nonlinear theory of oscillations of shallow shells). Dokl.Akad.Nauk SSSR, Vol.110, № 5, 1956.

15. Krasnosel'skii, M.A., Skhodimost' metoda Galerkina dlia nelineinykh uravnenii (Convergence of Galerkin's method for nonlinear equations). Dokl.Akad.Nauk SSSR, Vol.73, № 6, 1950.

16. Mikhlin, S.G., O metode Rittsa v nelineinykh zadachakh (On Ritz method in nonlinear problems). Dokl.Akad.Nauk SSSR, Vol.142, № 4, 1962.

17. Novozhilov, V.V., Ob odnom priblizhennom metode resheniia kraevykh zadach dlia differentsial'nykh uravnenii v obyknovennykh proizvodnykh) (Some approximate methods of solution of boundary-value problems for ordinary differential equations). *PMM* Vol.16, № 3, 1952.

18. Todorov, M., Annu.inzh.stroit.Inst., Sofia, Vol.6, Book 2, 1954.

19. Simeonov, V.L., Annu.inzh.stroit.Inst., Sofia, Vol.11, Book 1, 1959.

20. Goncharov, V.L., Teoriia interpolirovaniia i priblizheniia funktsii (Theory of Interpolation and Approximation of Functions). Gostekhizdat, 1954.

Translated by D.B.McV.